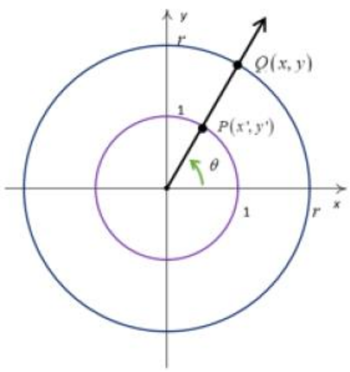
Beyond the Unit Circle

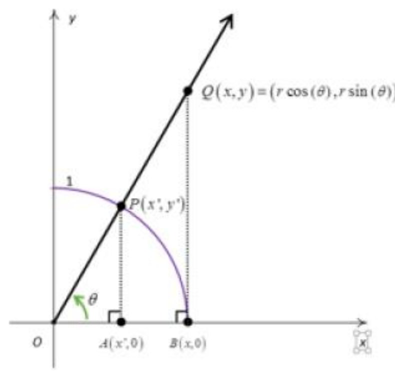
See Section 2.5 of the textbook (pages 95-101).

Recall that in defining the cosine and sine functions in **Section 2.2**, we assigned to each angle a position on the Unit Circle. Here we broaden our scope to include circles of radius *r* centered at the origin.

Determining Cosine and Sine

Consider for the moment the *acute* angle θ drawn below in standard position.





Let $Q(x, y)$ be the point on the terminal side of θ which lies on the circle $x^2 + y^2 = r^2$, and let $P(x', y')$ be the point on the terminal side of θ which lies on the Unit Circle. Now consider dropping perpendiculars from *P* and *Q* to create two triangles, ΔOPA and ΔOQB. These triangles are similar. Thus, it follows that $\frac{x}{x'} = \frac{r}{1} = r$, from which $x = rx'$. We similarly find $y = ry'$. Since, by definition $x' = \cos(\theta)$, and $y' = \sin(\theta)$, we get the coordinates of *Q* to be $x = r\cos(\theta)$ and $y = r\sin(\theta)$. By reflecting these points through the *x*-axis, *y*-axis and origin, we obtain the result for all non-quadrantal angles θ, and we leave it to the reader to verify these formulas hold for the quadrantal angles.

Not only can we describe the coordinates of *Q* in terms of $\cos(\theta)$ and $\sin(\theta)$, but since the radius of the circle is $r = \sqrt{x^2 + y^2}$, we can also express $\cos(\theta)$ and $\sin(\theta)$ in terms of the coordinates of *Q*. Throughout this textbook, by convention, the radius *r* of a circle is treated as positive as it relates to solving for trigonometric values.

These results are summarized in the following theorem.

Theorem 2.6. If $Q(x, y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2 + y^2 = r^2$ then $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Moreover,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}.$$

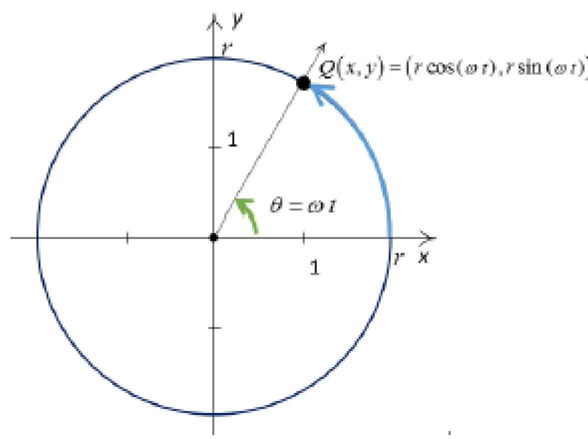
Note that in the case of the Unit Circle we have $r = \sqrt{x^2 + y^2} = 1$, so **Theorem 2.6** reduces to our Unit Circle definitions of $\cos(\theta)$ and $\sin(\theta)$.

Examples 2.5.1

Suppose that the terminal side of an angle θ, when plotted in standard position, contains the point $Q(3,-4)$. Find $\cos(\theta)$ and $\sin(\theta)$.

In **Chapter 1** we approximated the radius of the circle of revolution at 40.7608° North Latitude on Earth to be 2999 miles. Justify this approximation if the radius of the circle of revolution at the Equator is approximately 3960 miles.

Position of a Particle in Circular Motion



Theorem 2.6 gives us what we need to describe the position of an object traveling in a circular path of radius *r* with constant angular velocity ω. Suppose that at time *t*, the object has swept out an angle measuring θ radians. If we assume that the object is at the point $(r,0)$ when $t=0$, the angle θ is in standard position.

By definition, $\omega=\dfrac{\theta}{t}$, which we rewrite as $\theta=\omega t$.

According to **Theorem 2.6**, the location of the object $Q(x,y)$ on the circle is found using the equations $x=r\cos(\theta)$ and $y=r\sin(\theta)$. With $\theta=\omega t$, we have $x=r\cos(\omega t)$ and $y=r\sin(\omega t)$. Hence, at time *t*, the object is at the point $(r\cos(\omega t), r\sin(\omega t))$.

We have just argued the following:

Equations for Circular Motion: Suppose an object is traveling on a circular path of radius r centered at the origin with constant angular velocity ω. If $t=0$ corresponds to the point $(r,0)$, then the x and y coordinates of the object are functions of t and are given by $x=r\cos(\omega t)$ and $y=r\sin(\omega t)$. Here, $\omega>0$ indicates a clockwise direction.

Example 2.5.2

Salt Lake Community College is at approximately 40.7608° North Latitude. Find the equations of motion of Salt Lake Community College as the Earth rotates.

Determining the Other Four Circular Functions

We have generalized the cosine and sine functions from coordinates on the Unit Circle to coordinates on circles of radius *r*. Using **Theorem 2.6** in conjunction with **Theorem 2.3**, we generalize the remaining circular functions in kind.

Theorem 2.7. Suppose $Q(x, y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2 + y^2 = r^2$. Then the circle has radius r and

- $\csc(\theta) = \dfrac{r}{y} = \dfrac{\sqrt{x^2 + y^2}}{y}$, provided $y \neq 0$.
- $\sec(\theta) = \dfrac{r}{x} = \dfrac{\sqrt{x^2 + y^2}}{x}$, provided $x \neq 0$.
- $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

Example 2.5.3

Suppose the terminal side of θ, when plotted in standard position, contains the point $Q(4, -2)$. Find the values of the six circular functions of θ.

Suppose θ is a Quadrant IV angle with $\cot(\theta) = -4$. Find the values of the five remaining circular functions of θ.